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# A Hopf algebra structure in self-dual gravity

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## ABSTRACT

The two-dimensional non-linear sigma model approach to Self-dual Yang-Mills theory and to Self-dual gravity given by Q-Han Park is an example of the deep interplay between two and four dimensional physics. In particular, Husain's two-dimensional chiral model approach to Self-dual gravity is studied. We show that the infinite hierarchy of conservation laws associated to the Husain model carries implicitly a hidden infinite Hopf algebra structure.

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## 1. Introduction

Self-dual gravity (SdG) is a very interesting arena to understand the interplay between the physics and/or mathematics in two and four dimensions [1,2,3]. Ooguri and Vafa have shown that SdG is actually an effective theory of  $N=2$  Strings, and that it has a most natural interpretation in the context of stringy physics [4]. The  $N=2$  Heterotic String Theory (SdG coupled with Self-dual Yang-Mills theory (SdYM)) can be seen as the “master system” in which one is able to understand the deep interrelation between the integrability in four dimensions (coming from the Penrose construction [5]) and the integrability arising in the two dimensional models [4]. It is hoped that the *quantum geometry* (geometry associated to the relevant  $N = 2$  superconformal field theory) coming from  $N=2$  Heterotic Superstrings will generalize the Ricci-flat Kähler geometry involved in SdG, since SdG is conjectured to be related in a deep way to the quantum geometry through the  $N=2$  supersymmetric non-linear sigma models on Calabi-Yau manifolds [4] (for a recent review about quantum geometry see [6]). This quantum geometry might be useful

in order to understand (at the algebraic geometry level) the relation of the two twistor constructions associated with SdG and SdYM. This is because many of the features of the *classical* geometry are valid at quantum level.

All string theories are Conformal Field Theories in two dimensions (CFT2's). Almost all the ‘magic’ properties of string theory have their origin in this CFT2. These symmetries are expressed through the Virasoro, Affine Lie and  $W_\infty$ -algebras and they can be very useful in inducing (up to dimensional obstructions) nice properties to the self-dual structures in four dimensions. One of these properties is the existence of conserved quantities in a physical theory. In a CFT2 (due to the infinite dimensional symmetries) always arise infinite hierarchies of conserved quantities.

On the other hand in SdG the history has been a bit different. The first notion of an infinite hierarchy of conserved quantities has been given in a paper by Boyer and Plebański [7]. Using the first and the second heavenly equations

[8] it is shown that SdG admits an infinite hierarchy of conservation laws. These quantities are defined in terms of the first and second key holomorphic functions  $\Omega$  and  $\Theta$ . Later, some global aspects associated to the above construction have been studied [9]. For this the maximal isotropic submanifolds formalism is employed. This construction is formal, thus avoiding the use of the infinitesimal deformation of the twistor space [5]. The symmetries play an important role here, showing that the underlying symmetry group is the area preserving diffeomorphisms of a twistor surface (totally geodesic null surface). In fact, this approach shows the existence of a correspondence between the formal holomorphic bundles over the Riemann sphere  $\mathbf{CP}^1$  and the group of area preserving diffeomorphisms. After this, Takasaki, in a series of papers [10], shows the existence of a hyper-Kähler infinite hierarchy. He found that it was possible to construct inequivalent metrics in SdG by using the area preserving diffeomorphisms group.

Later, Strachan has shown that the existence of the infinite hierarchy is related with an infinite family of twistor surfaces [11]. To be more precise, he found a one to one correspondence between a family of twistor surfaces and the conserved charges. Similarly to [7,9], Strachan starts also from the heavenly equation to make his construction.

Another approach to this setting was given recently by Husain [12]. He has used strongly his own result concerning the use of the equivalence of SdG and the two-dimensional chiral model with the gauge group defined by the group of area preserving diffeomorphism of an “internal” two-surface  $\mathcal{N}^2$  [13]. Husain’s construction of the infinite hierarchy involves the use of induced properties of the two-dimensional chiral model to SdG. In the present paper we continue this philosophy and explore how some other features of a more general two-dimensional field theory (i.e., a CFT2) can be carry over to self-dual structures in four-dimensions.

In Sect. 2 we briefly review Husain’s construction of the infinite hierarchy in SdG [12,14]. After this in Sect. 3 we show how this hierarchy has associated a hidden infinite Hopf algebra structure. Finally, in Sect. 4 our final remarks are

given.

## 2. Infinite Hierarchy of Conserved Currents in Self-dual Gravity

### 2.1. CONSERVATION LAWS FROM HUSAIN'S CHIRAL MODEL

In this section we shall briefly review the necessary arguments in order to display the Hopf algebra structure of  $\text{sdiff}(\mathcal{N}^2)$  in the next section. In Ref. [12,13], starting from the Ashtekar-Jacobson-Smolín (AJS) formulation for SdG [15], Husain found a set of equations for four vector fields  $\mathcal{U}$ ,  $\mathcal{V}$ ,  $\mathcal{X}$  and  $\mathcal{T}$ . These vector fields arise after a light-cone variable decomposition of  $V_0$  and a triad of vector fields  $V_i$ ,  $i = 1, 2, 3$  on a three manifold  $\Sigma^3$ .  $V_0$  is a vector field used in the  $3 + 1$  splitting. Here  $\Sigma^3$  comes from the global splitting of the space-time manifold  $\mathcal{M}$  into  $\Sigma^3 \times \mathbb{R}$ . The four-manifold  $\mathcal{M}$  admits local coordinates  $\{x^0, x^1, x^2, x^3\}$ . In fact for the most relevant part of Husain's construction the three manifold  $\Sigma^3$  can be identified locally with  $\mathbb{R}^3$ . Choosing suitable expressions for the vector fields  $\mathcal{U}$ ,  $\mathcal{V}$ ,  $\mathcal{X}$  and  $\mathcal{T}$ , Husain proved that the AJS equations led to the two-dimensional chiral model (with the two-dimensional 'space-time' ( $\mathbb{R}^2$ ) coordinates  $\{x, t\}$  and with gauge group being the group of area-preserving diffeomorphisms of a two-dimensional 'internal space'  $\mathcal{N}^{2*}$  with local coordinates  $\{p, q\}$ ). This chiral model is given by the equations

$$f_{01} = \partial_0 a_1 - \partial_1 a_0 + \{a_0, a_1\}_P = 0, \quad (2.1a)$$

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★ As we make only local considerations we assume the space  $\mathcal{N}^2$  to be a two-dimensional simply connected symplectic manifold with local coordinates  $p$  and  $q$ . This space has a natural local symplectic structure given by the local area form  $\omega = dp \wedge dq$ . The group  $\text{SDiff}(\mathcal{N}^2)$  is precisely the group of diffeomorphisms on  $\mathcal{N}^2$  preserving the symplectic structure  $\omega$ , i.e. for all  $g \in \text{SDiff}(\mathcal{N}^2)$ ,  $g^*(\omega) = \omega$ .

$$\partial_0 a_0 + \partial_1 a_1 = \partial_i a_i = 0, \quad (2.1b)$$

where  $i = 0, 1$  (or  $x, t$ ),  $\{, \}_P$  means the Poisson bracket in  $p$  and  $q$  and  $a_i = a_i(x, t, p, q)$  are analytic functions on  $\mathcal{Y} = \mathbb{R}^2 \times \mathcal{N}^2$  which satisfy the above equations. In the Husain's formulation the functions  $a_i$ 's are close related to the hamiltonian vector fields  $\mathcal{U}^a$  and  $\mathcal{V}^a$  which are given by

$$\mathcal{U}^a = \left( \frac{\partial}{\partial t} \right) + \omega^{ba} \partial_b H_0, \quad (2.2a)$$

$$\mathcal{V}^a = \left( \frac{\partial}{\partial x} \right) + \omega^{ba} \partial_b H_1, \quad (2.2b)$$

where  $H_0, H_1 \in C^\infty(\mathcal{Y})$  are the hamiltonian functions which differ from  $a_0$  and  $a_1$  by the arbitrary functions  $G$  and  $F$  respectively [12]

$$a_0 = H_0 + G, \quad a_1 = H_1 - F. \quad (2.3)$$

$\omega^{ab} = \left( \frac{\partial}{\partial p} \right)^{[a} \otimes \left( \frac{\partial}{\partial q} \right)^{b]}$  whose inverse is precisely the symplectic local form  $\omega_{ab} = dp_a \wedge dq_b$  on  $\mathcal{N}^2$  and satisfying the relation  $\omega^{ab} \omega_{bc} = \delta_c^a$ . In all that  $a, b = p, q$ . All these considerations are of course local<sup>†</sup>.

Now we define a two-dimensional vector field-valued 1-form with ‘space-time’ components precisely the vector fields  $\mathcal{U}$  and  $\mathcal{V}$ , that is

$$\mathcal{A}_i = (\mathcal{U}dt + \mathcal{V}dx)_i. \quad (2.4)$$

where the notation means  $\mathcal{A}_0 = \mathcal{A}_t = \mathcal{U}$  and  $\mathcal{A}_1 = \mathcal{A}_x = \mathcal{V}$ .

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<sup>†</sup> Globally the symplectic form is defined by  $\omega : T\mathcal{N}^2 \rightarrow T^*\mathcal{N}^2$  and inverse  $\omega^{-1} : T^*\mathcal{N}^2 \rightarrow T\mathcal{N}^2$ . While the hamiltonian vector fields are  $\mathcal{U}_{H_i} = \omega^{-1}(dH_i)$  satisfying the algebra  $[\mathcal{U}_{H_i}, \mathcal{U}_{H_j}] = \mathcal{U}_{\{H_i, H_j\}}$  where  $\{, \}$  stands for the Poisson bracket. Locally it can be written as  $\{H_i, H_j\} = \omega^{-1}(dH_i, dH_j) = \omega^{ab} \partial_a H_i \partial_b H_j$ .

Thus the 1-form  $\mathcal{A} = \mathcal{A}_i dx^i$  can be interpreted as a  $\text{sdiff}(\mathcal{N}^2)$ -valued connection 1-form on  $\mathfrak{R}^2$ , *i.e.*  $\mathcal{A} \in C^\infty(T^*\mathfrak{R}^2 \otimes \text{sdiff}(\mathcal{N}^2))$ . This connection is of course *flat* and that condition implies that its curvature  $\Omega$  vanishes

$$\Omega = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0. \quad (2.5)$$

Locally this condition yields

$$\mathcal{F}_{01} = \partial_0 \mathcal{A}_1 - \partial_1 \mathcal{A}_0 + [\mathcal{A}_0, \mathcal{A}_1] = 0, \quad (2.6a)$$

where  $[\cdot, \cdot]$  stands for the Lie bracket.

According to the Ref. [12] the vector fields  $\mathcal{U}$  and  $\mathcal{V}$  must satisfy also the relation

$$\frac{\partial \mathcal{U}}{\partial t} + \frac{\partial \mathcal{V}}{\partial x} = 0$$

or

$$\partial_i \mathcal{A}_i = \partial_0 \mathcal{A}_0 + \partial_1 \mathcal{A}_1 = 0. \quad (2.6b)$$

Thus in terms of the connection  $\mathcal{A}_i$ , the chiral Eqs. (2.1a, b) can be expressed as Eqs. (2.6a, b) respectively.

The first conserved current is taken to be

$$\mathcal{J}_i^{(1)}(x, t, p, q) := \mathcal{A}_i(x, t, p, q), \quad (2.7)$$

which is immediately seen to be conserved using the equation of motion (2.6b). Introducing the two-dimensional Levi-Civita symbol  $\epsilon_{ij}$  ( $\epsilon_{01} = -\epsilon_{10} = 1$ ), it is easy to see that the first conserved current is

$$\mathcal{J}_i^{(1)} = \epsilon_i^j \partial_j \eta^{(1)}, \quad (2.8)$$

implying the existence of a vector field  $\eta^{(1)}$ .

The second conserved current is defined by

$$\mathcal{J}_i^{(2)} := [\mathcal{A}_i, \eta^{(1)}]. \quad (2.9)$$

This current will be conserved using the Eqs. (2.6a, b). Similarly to the above equation the  $n$ -th conserved current can be defined to be

$$\mathcal{J}_i^{(n)} := [\mathcal{A}_i, \eta^{(n-1)}], \quad (2.10)$$

which will be also conserved using the above argument.

Using now mathematical induction after several steps one can prove that the  $(n+1)$ -th current

$$\mathcal{J}_i^{(n+1)} := [\mathcal{A}_i, \eta^{(n)}], \quad (2.11)$$

is also conserved (for details see Ref. [12]).

In what follows we will use only the first conserved current. The consideration of higher order conserved currents remain to be addressed.

The conserved charges  $Q^{(1)}(t)$  and  $Q^{(2)}(t)$  can be defined as:

$$Q^{(1)}(t) = \int_{\mathbb{R} \times \mathcal{N}^2} d^2 \bar{s} \, dx \mathcal{J}_0^{(1)}(t, x, \bar{s}), \quad (2.12a)$$

where  $\bar{s}$  are the local coordinates on  $\mathcal{N}^2$  (*i.e.*  $d^2 \bar{s} = dp dq$ ) and  $\mathcal{J}_0^{(1)} = \mathcal{A}_0 = \mathcal{U}$ , then

$$Q^{(1)}(t) = \int_{\mathfrak{R} \times \mathcal{N}^2} d^2 \bar{s} \, dx \, \mathcal{U}(t, x, p, q) \quad (2.12b)$$

and

$$Q^{(2)}(t) = \int_{\mathfrak{R} \times \mathcal{N}^2} d^2 \bar{s} \, dx \, \mathcal{J}_0^{(2)}(t, x, \bar{s}), \quad (2.13a)$$

$$Q^{(2)}(t) = \int_{\mathfrak{R} \times \mathcal{N}^2} d^2 \bar{s} \, dx [\mathcal{U}, \int^x dx' \mathcal{U}(t, x', p, q)]. \quad (2.13b)$$

## 2.2. AFFINE LIE ALGEBRA ASSOCIATED TO THE LIE ALGEBRA OF $\text{SDIFF}(\mathcal{N}^2)$

In Ref. [14] Husain found an affine Lie algebra (of the Kac-Moody type) associated with the Lie algebra of area preserving diffeomorphisms. Beginning from the  $\text{sdiff}(\mathcal{N}^2)$ -chiral equations (2.1a, b) Husain show the existence of an infinite dimensional hidden non-local symmetry associated with the Poisson bracket. The conservation law

$$\partial_i J_i^{(n)}(x, t, p, q) = 0 \quad (2.14)$$

for the corresponding currents of Eq. (2.10)

$$J_i^{(n)}(x, t, p, q) = \epsilon_{ij} \partial_j \Lambda^{(n+1)}(x, t, p, q) \quad (2.15)$$

determines the existence of a scalar function  $\Lambda^{(n+1)}$  on  $\mathcal{Y} = \mathfrak{R}^2 \times \mathcal{N}^2$ . This function can be obtained from a hierarchy of such a functions by



$$\Lambda^{(n+1)} = \int_{-\infty}^x dx' D_0 \Lambda^{(n)}(t, x') \quad (2.16)$$

where  $D_0$  is the zero component of the covariant derivative  $D_i \Lambda := \partial_i \Lambda + \{a_i, \Lambda\}_P$ .

The currents (2.15) are conserved under the symmetry transformations

$$\delta_\Lambda a_i = D_i \Lambda. \quad (2.17)$$

Now, in order to obtain the above mentioned Kac-Moody algebra Husain defines the generators associated to the transformation (2.17) by

$$T^{(n)} := \int dt dx (\delta_\Lambda^{(n)} a_i) \frac{\delta}{\delta a_i}$$

or

$$= \int dt dx (D_i \Lambda^{(n)}) \frac{\delta}{\delta a_i}. \quad (2.18)$$

We first wish to give some general remarks in order to fix the notation for further considerations. For this we will use the standard notation of general 2-index infinite algebras [16,17,18]. To be more precise let  $\{\mathbf{e}_\mathbf{m}(\mathbf{x})\}$  be a generic basis of hamiltonian functions satisfying the algebra

$$\{\mathbf{e}_\mathbf{m}(\mathbf{x}), \mathbf{e}'_\mathbf{m}(\mathbf{x})\} = C_{\mathbf{m}\mathbf{m}'}^{\mathbf{m}''} \mathbf{e}''_{\mathbf{m}}(\mathbf{x}). \quad (2.19)$$

Expanding now  $\Lambda^{(n)}(x, t, p, q)$  in the above basis  $\{\mathbf{e}_\mathbf{m}(\mathbf{x})\}$  we have

$$\Lambda^{(n)}(x, t, p, q) = \sum_{\mathbf{m}} \mathbf{e}_{\mathbf{m}}(\mathbf{x}) \Lambda_{\mathbf{m}}^{(n)}(x, t) \quad (2.20)$$

where  $\mathbf{m}, \mathbf{m}'$  and  $\mathbf{m}''$  are constant 2-vectors, (*i.e.*  $\mathbf{m} = (m_1, m_2)$ , with  $m_1, m_2 \in \mathbf{Z}$  and  $\mathbf{x} = (p, q)$  is a 2-vector with  $p, q$  the local coordinates on  $\mathcal{N}^2$ ) and  $C_{\mathbf{m}\mathbf{m}'}^{\mathbf{m}''}$  are the structure constants which depend on the topology of  $\mathcal{N}^2$ .

Let  $\mathcal{L}_{\mathbf{e}_{\mathbf{m}}(\mathbf{x})} \equiv \mathcal{L}_{\mathbf{m}}$  be the associated hamiltonian vector fields which satisfy the Poisson algebra  $\text{sdiff}(\mathcal{N}^2)$

$$[\mathcal{L}_{\mathbf{m}}, \mathcal{L}_{\mathbf{m}'}] = C_{\mathbf{m}\mathbf{m}'}^{\mathbf{m}''} \mathcal{L}_{\mathbf{m}''}. \quad (2.21)$$

Any general function  $\mathcal{F}(\mathbf{x})$  can be expressed as the linear combination of the basis of the vector fields

$$\mathcal{F}(\mathbf{x}) = \sum_{\mathbf{m}} f_{\mathbf{m}} \mathcal{L}_{\mathbf{m}}, \quad (2.22)$$

where  $f_{\mathbf{m}}$  are the expansion coefficients.

In terms of the basis  $\{\mathbf{e}_{\mathbf{m}}(\mathbf{x})\}$  the generators  $T^{(n)}$  can be expressed by

$$T_{\mathbf{m}}^{(n)} = \int dt dx (D_i \Lambda_{\mathbf{m}}^{(n)}) \frac{\delta}{\delta a_i}. \quad (2.23)$$

Finally Husian found also an affine Lie algebra structure for these generators  $T_{\mathbf{m}}^{(n)}$  to be

$$[T_{\mathbf{m}}^{(m)}, T_{\mathbf{m}'}^{(n)}] = C_{\mathbf{m}\mathbf{m}'}^{\mathbf{m}''} T_{\mathbf{m}''}^{(m+n)}. \quad (2.24)$$

The most trivial case,  $m = n = 0$ , corresponds precisely with the Poisson algebra  $\text{sdiff}(\mathcal{N}^2)$  (2.21)

$$[T_{\mathbf{m}}^{(0)}, T_{\mathbf{m}'}^{(0)}] = C_{\mathbf{m}\mathbf{m}'}^{\mathbf{m}''} T_{\mathbf{m}''}^{(0)}. \quad (2.25)$$

These results are of course at the classical level. The quantization might be related to knot theory and integrable 2d field theory. According with Ref. [14] it is possible that this connection comes from the Yang-Baxter equation.

### 3. A Hopf Algebra Structure in Self-dual Gravity

In this section we make the construction of the Hopf algebra  $\mathcal{H}$  associated to the affine Lie algebra (2.24), in particular, that associated to the Poisson algebra  $\text{sdiff}(\mathcal{N}^2)$ .

We define new generators (or charges)  $\mathcal{Q}_{\mathbf{m}}^{(n)}(t)$  from Eq. (2.23) by

$$T_{\mathbf{m}}^{(n)} = \int dt \mathcal{Q}_{\mathbf{m}}^{(n)}(t) \quad (3.1)$$

where

$$\mathcal{Q}_{\mathbf{m}}^{(n)}(t) = \int_{-\infty}^{\infty} dx j_{\mathbf{m}}^{(n)}(x, t) \quad (3.2)$$

and

$$j_{\mathbf{m}}^{(n)}(x, t) = D_i \Lambda_{\mathbf{m}}^{(n)}(x, t) \frac{\delta}{\delta a_i}. \quad (3.3)$$

The defined charges satisfy of course the algebra

$$[\mathcal{Q}_{\mathbf{m}}^{(m)}(t), \mathcal{Q}_{\mathbf{m}'}^{(n)}(t')] = \delta(t - t') C_{\mathbf{m}\mathbf{m}'}^{\mathbf{m}''} \mathcal{Q}_{\mathbf{m}''}^{(m+n)}(t). \quad (3.4)$$

Now we decompose the integral (3.2) into two integrals

$$\mathcal{Q}_{\mathbf{m}}^{(n)}(t) = \int_{-\infty}^0 dx j_{\mathbf{m}}^{(n)}(x, t) + \int_0^{\infty} dx j_{\mathbf{m}}^{(n)}(x, t). \quad (3.5)$$

Thus we can write the last equation as

$$\mathcal{Q}_{\mathbf{m}}^{(n)}(t) = \mathcal{Q}_{\mathbf{m}+}^{(n)}(t) + \mathcal{Q}_{\mathbf{m}-}^{(n)}(t). \quad (3.6)$$

where the signs  $+(-)$  correspond to positive (negative) values of  $x$ .

In what follows we restrict ourselves to the case (2.25) (*i.e*  $m = n = 0$ ), but the generalization for the general case is easy to get.

### 3.1. THE HOPF ALGEBRA STRUCTURE

It is well known that the diffeomorphism group  $\text{Diff}(M)$  of a compact manifold  $M$  is not a Banach Lie Group but a Hilbert manifold [19,20]. Due to that  $\mathcal{N}^2$  is a compact symplectic manifold with symplectic form  $\omega = dp \wedge dq$  which defines a subgroup  $\text{SDiff}(\mathcal{N}^2)$  of  $\text{Diff}(\mathcal{N}^2)$ . The associated Lie algebra to  $\text{SDiff}(\mathcal{N}^2)$  is  $\text{sdiff}(\mathcal{N}^2)$ , the space of locally hamiltonian vector fields on  $\mathcal{N}^2$ . This algebra has the structure of a Frechet manifold and therefore the product  $\text{sdiff}(\mathcal{N}^2) \times \text{sdiff}(\mathcal{N}^2)$  is the product of Frechet manifolds which is also Frechet [20].

Consider the infinite dimensional Lie algebra  $\text{sdiff}(\mathcal{N}^2)$  over the field  $\mathfrak{R}$  and a basis of this algebra to be the  $\mathcal{Q}_{\mathbf{m}}^{(0)}(t)$ . Introducing an infinite dimensional Universal Enveloping algebra  $\mathcal{H} = \mathbf{U}(\text{sdiff}(\mathcal{N}^2))$ , we can now define on  $\mathcal{H}$  a structure of a Hopf algebra (for details see Ref. [21]). Thus, in what follows the tensorial product of Universal Enveloping algebras of  $\text{sdiff}(\mathcal{N}^2)$ ,  $\mathcal{H} \otimes \mathcal{H}$ , will be defined in the Frechet sense.

Following MacKay's [22] we define the co-product,  $\Delta$ , as follows:

$$\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}, \quad (3.7)$$

that means,

$$\mathcal{Q}_{\mathbf{m}}^{(0)} \mapsto \Delta(\mathcal{Q}_{\mathbf{m}}^{(0)}) = \mathcal{Q}_{\mathbf{m}}^{(0)} \otimes 1 + 1 \otimes \mathcal{Q}_{\mathbf{m}}^{(0)}. \quad (3.8)$$

On the identity element,  $1 \in \mathcal{H}$ , the co-product is defined by

$$\Delta(1) = 1 \otimes 1. \quad (3.9)$$

This co-product  $\Delta$  is an  $\mathfrak{R}$ -algebra homomorphism. The definition for the co-product, Eq. (3.8), is fulfilled also when we define

$$\mathcal{Q}_{\mathbf{m}}^{(0)}(t) = \mathcal{Q}_{\mathbf{m} \ 1+}^{(0)}(t) \otimes \mathcal{Q}_{\mathbf{m} \ 2-}^{(0)}(t), \quad (3.10)$$

where the signs  $+$  and  $-$  correspond to the decomposition given in Eq. (3.6), the numbers 1,2 mean the first and the second entries of  $\mathcal{H} \otimes \mathcal{H}$ , respectively.

Defining now the ‘twist’ map  $\tau : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$  given by  $\tau(\mathcal{Q}_{\mathbf{m}}^{(0)} \otimes \mathcal{Q}_{\mathbf{m}'}^{(0)}) = \mathcal{Q}_{\mathbf{m}'}^{(0)} \otimes \mathcal{Q}_{\mathbf{m}}^{(0)}$ , one can see that the co-product (3.8) is co-commutative. This holding because the relation  $\tau \circ \Delta = \Delta$  is fulfilled.

With the the above definition one can show that the coproduct satisfies the co-associativity axiom

$$(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta. \quad (3.11)$$

where  $id$  is the identity map, *i.e.*  $id : \mathcal{H} \rightarrow \mathcal{H}$ ,  $id(\mathcal{Q}_{\mathbf{m}}^{(0)}(t)) = \mathcal{Q}_{\mathbf{m}}^{(0)}(t)$ . To prove this axiom one can decompose the charge  $\mathcal{Q}_{\mathbf{m}}^{(0)}(t)$  into three parts just as it is mentioned in Ref. [22].

On the other hand the co-unit  $\epsilon$  is also an  $\mathfrak{R}$ -algebra homomorphism  $\epsilon : \mathcal{H} \rightarrow \mathfrak{R}$ , which one can define by

$$\epsilon(\mathcal{Q}_{\mathbf{m}}^{(0)}(t)) = 0, \quad \epsilon(1) = 1. \quad (3.12)$$

where  $0 \in \mathfrak{R}$ . With the above definitions for the co-product and co-unit one can easily prove that the co-unit axiom is fulfilled.

$$(id \otimes \epsilon) \circ \Delta = (\epsilon \otimes id) \circ \Delta. \quad (3.13)$$

The antipode is an  $\mathfrak{R}$ -algebra antihomomorphism  $S : \mathcal{H} \rightarrow \mathcal{H}$ . In our case we define the antipode as

$$S(\mathcal{Q}_{\mathbf{m}}^{(0)}(t)) = -\mathcal{Q}_{\mathbf{m}}^{(0)}(-t). \quad (3.14)$$

It is an easy matter to see that the definitions (3.8) and (3.14) satisfy the axiom of the antipode:

$$m \circ (S \otimes id) \circ \Delta = m \circ (id \otimes S) \circ \Delta, \quad (3.15)$$

where  $m$  is the operation product in  $\mathcal{H}$  and is defined as a homomorphism  $m : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H}$ ,  $m(\mathcal{Q}_{\mathbf{m}}^{(0)} \otimes \mathcal{Q}_{\mathbf{m}'}^{(0)}) := [\mathcal{Q}_{\mathbf{m}}^{(0)}, \mathcal{Q}_{\mathbf{m}'}^{(0)}] = C_{\mathbf{m}\mathbf{m}'}^{\mathbf{m}''} \mathcal{Q}_{\mathbf{m}''}^{(0)}$ , where  $\mathcal{Q}^{(1)\mathbf{m}}, \mathcal{Q}^{(1)\mathbf{m}'} \in \mathcal{H}$

From the point of view of current algebra the above definition for the antipode map, (3.14), corresponds to a usual, classical, **PT** transformation

$$S(j_{\mathbf{m}}^{(0)}(x, t)) = -j_{\mathbf{m}}^{(0)}(-x, -t). \quad (3.16)$$

which coincides with McKay's result [22] whenever

$$S(\Lambda_{\mathbf{m}}^{(n)}(x, t)) = \Lambda_{\mathbf{m}}^{(n)}(-x, -t). \quad (3.16)$$

## 4. Final Remarks

In the present paper we have used the infinite hierarchy of conserved quantities for SdG just as it has been given by Husain [12,14]. Then using the conserved charges we display how they possess a Hopf algebra structure. Many interesting implications might be derived from our results. First of all one could generalize every thing presented here by considering the Moyal algebra. This is the unique deformation (with deformation parameter to be  $k$ ) of the Poisson algebra considered here  $\text{sdiff}(\mathcal{N}^2)$ . The Moyal algebra is

$$[\mathcal{L}_{\mathbf{m}}, \mathcal{L}_{\mathbf{m}'}] = \frac{1}{k} \text{Sin}\left(k\mathbf{m} \times \mathbf{m}'\right) \mathcal{L}_{\mathbf{m}+\mathbf{m}'}. \quad (4.1)$$

One would like to find the Hopf algebra associated to the Moyal deformation of  $\text{sdiff}(\mathcal{N}^2)$ . In fact there exist a further  $q$ -deformation of the Moyal algebra. It was considered in Refs. [23,24,25] a  $q$ -deformation of the Moyal algebra

$$[\mathcal{L}_{\mathbf{m}}, \mathcal{L}_{\mathbf{m}'}]_{q^{\mathbf{m} \times \mathbf{m}'}} = \left(p^{\mathbf{m} \times \mathbf{m}'} - p^{-\mathbf{m} \times \mathbf{m}'}\right) \mathcal{L}_{\mathbf{m}+\mathbf{m}'}. \quad (4.2)$$

This structure of quantum algebra of the Moyal deformation of  $\text{sdiff}(\mathcal{N}^2)$  might be important in order to give a bit of more consistency to our results [26].

Another way to address a quantum algebra structure is trough the existence of a QUE (Quantized Universal Enveloping)-algebra associated with both  $\text{sdiff}(\mathcal{N}^2)$  and its Moyal deformation might be achieved using the methods of massive 2d quantum field theory given by Le Clair and Smirnov [27,28]. Then by considering these methods one will have a no(co)commutative (co)product which give us directly a  $q$ -deformed version of  $\text{sdiff}(\mathcal{N}^2)$  and its Moyal deformation.

One more possible things to be considered in this context concerning to conserved currents. Strachan found an infinite hierarchy of symmetries associated to SdG equations [29]. A different approach has been given also very recently by Husain [14]. It would be interesting to make the connection between both approaches which seem to be equivalent.

Furthermore, since the Husain's model has been solved (for finite subgroups of  $\text{sdiff}(\mathcal{N}^2)$ ) in terms of harmonic maps [30], one could ask about the possibility to address the whole problem concerning conserved laws, Hopf algebras, its deformations and harmonic maps.

Finally, given the possible relations between self-dual gravity and knot theory by using the Yang-Baxter equation one could to use the technique [31] in order to get such a connections. This technique has ben succesful to find solutions in self-dual gravity and to obtain a like-WZW action which might be connected to 2d integrable field theory. We would like to address this considerations in a forthcoming paper.

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